RECOGNISING ACHIEVEMENT

GCE

Mathematics

Advanced GCE

Unit **4726:** Further Pure Mathematics 2

Mark Scheme for June 2011

OCR (Oxford Cambridge and RSA) is a leading UK awarding body, providing a wide range of qualifications to meet the needs of pupils of all ages and abilities. OCR qualifications include AS/A Levels, Diplomas, GCSEs, OCR Nationals, Functional Skills, Key Skills, Entry Level qualifications, NVQs and vocational qualifications in areas such as IT, business, languages, teaching/training, administration and secretarial skills.

It is also responsible for developing new specifications to meet national requirements and the needs of students and teachers. OCR is a not-for-profit organisation; any surplus made is invested back into the establishment to help towards the development of qualifications and support which keep pace with the changing needs of today's society.

This mark scheme is published as an aid to teachers and students, to indicate the requirements of the examination. It shows the basis on which marks were awarded by Examiners. It does not indicate the details of the discussions which took place at an Examiners' meeting before marking commenced.

All Examiners are instructed that alternative correct answers and unexpected approaches in candidates' scripts must be given marks that fairly reflect the relevant knowledge and skills demonstrated.

Mark schemes should be read in conjunction with the published question papers and the Report on the Examination.

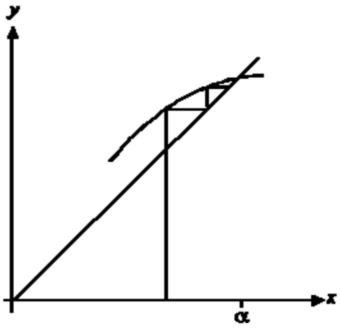
OCR will not enter into any discussion or correspondence in connection with this mark scheme.

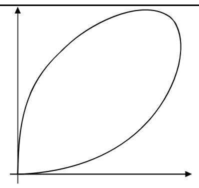
Any enquiries about publications should be addressed to:

OCR Publications
PO Box 5050
Annesley
NOTTINGHAM
NG15 0DL

Telephone: 0870 770 6622
Facsimile: 01223 552610
E-mail: publications@ocr.org.uk

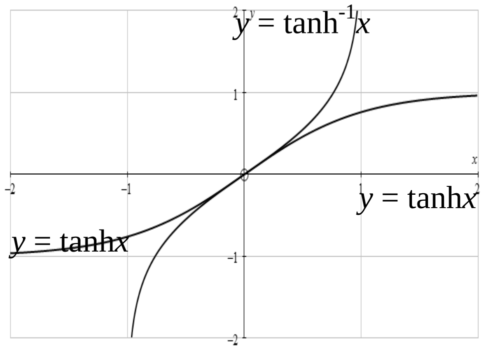
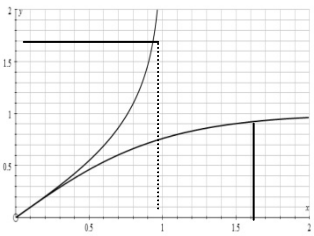
1	$\dfrac{2x+3}{(x+3)(x^2+9)} \equiv \dfrac{A}{x+3}+\dfrac{Bx+C}{x^2+9}$	**B1**	For correct form seen anywhere with letters or values
	$A=-\dfrac{1}{6}$	**B1**	For correct *A* (cover up or otherwise)
	$2x+3 \equiv A(x^2+9)+(Bx+C)(x+3)$	**M1**	For equating coefficients at least once.(or substituting values) into correct identity.
	$B=\dfrac{1}{6},\quad C=\dfrac{3}{2}$	**A1**	For correct *B* and *C*
	$\Rightarrow \dfrac{-1}{6(x+3)}+\dfrac{x+9}{6(x^2+9)}$	**A1**	For correct final statement cao, oe
		5	
2(i)	Asymptote $x = 2$	**B1**	For correct equation
	$y=x-4-\dfrac{13}{x-2}$	**M1**	For dividing out (remainder not required)
	$\Rightarrow$ asymptote $y=x-4$	**A1**	For correct equation of asymptote (ignore any extras)
		3	
(ii)	METHOD 1		**N.B. answer given**
	$x^2-(y+6)x+(2y-5)=0$	**M1**	For forming quadratic in *x*
	$b^2-4ac\left(\ge 0\right) \Rightarrow \left(y+6\right)^2-4(2y-5)\left(\ge 0\right)$	**M1**	For considering discriminant
	$\Rightarrow y^2+4y+56\left(\ge 0\right)$	**A1**	For correct simplified expression in *y* **soi**
	$\Rightarrow \left(y+2\right)^2+52\ge 0:$ this is true $\forall y$ So y takes all values	**A1**	For completing square (or equivalent) and correct conclusion **www**
	METHOD 2		
	Obtain $\dfrac{dy}{dx}=\dfrac{x^2-4x+17}{\left(x-2\right)^2}$ OR $1+\dfrac{13}{\left(x-2\right)^2}$	**M1**	For finding $\dfrac{dy}{dx}$ either by direct differentiation or dividing out first
		A1	For correct expression **oe.**
	$\Rightarrow \dfrac{dy}{dx}\ge 1\ \forall x,$	**M1**	For drawing a conclusion
	so y takes all values.	**A1**	For correct conclusion **www**
		4	
	Alternate scheme: Sketching graph		
	Graph correct approaching asymptotes from both side	**B1**	A graph with no explanation can only score 2
	Graph completely correct	**B1**	
	Explanation about no turning values	**B1**	
	Correct conclusion	**B1**	

3(i)	$x_1 = 3.1 \Rightarrow x_2 = 3.13140$, $x_3 = 3.14148$	**B1** **B1** **2**	For correct x_2 For correct x_3
(ii)	$F'(\alpha) \approx \frac{e_3}{e_2} = \frac{0.00471}{0.01479} = 0.318\ (0.31846)$ $F'(\alpha) = \frac{1}{\alpha} = 0.3178\ (0.31784)$	**M1** **A1** **B1** **3**	For dividing e_3 by e_2 For estimate of $F'(\alpha)$ For true $F'(\alpha)$ obtained from $\frac{d}{dx}(2+\ln x)$ **TMDP anywhere in (i) (ii) deduct 1 once (but answers must round to given values or A0)**
(iii)	 Staircase	**B1** **B1** **B1** **3**	For $y = x$ and $y = F(x)$ drawn, crossing as shown For lines drawn to illustrate iteration (Min 2 horizontal and 2 vertical seen) For stating "staircase"

4(i)	$x = r\cos\theta,\ y = r\sin\theta$ $\Rightarrow r = \dfrac{a\cos\theta\sin\theta}{\cos^3\theta + \sin^3\theta}$ for $0 \le \theta \le \frac{1}{2}\pi$	**M1** **A1** **A1** **3**	For substituting for *x* and *y* For correct equation **oe** (Must be $r = \ldots..$) For correct limits for θ (Condone <)
(ii)	$f\left(\frac{1}{2}\pi - \theta\right) = \dfrac{a\cos\left(\frac{1}{2}\pi - \theta\right)\sin\left(\frac{1}{2}\pi - \theta\right)}{\cos^3\left(\frac{1}{2}\pi - \theta\right) + \sin^3\left(\frac{1}{2}\pi - \theta\right)}$ $= \dfrac{a\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}$ $f(\theta) = f\left(\frac{1}{2}\pi - \theta\right) \Rightarrow \alpha = \frac{1}{4}\pi$	**M1** **A1** **A1** **3**	**N.B. answer given** For replacing θ by $\left(\frac{1}{2}\pi - \theta\right)$ in their $f(\theta)$ For correct simplified form. (Must be convincing) For correct reason for $\alpha = \frac{1}{4}\pi$
(iii)	$r = \dfrac{a.\frac{1}{\sqrt{2}}.\frac{1}{\sqrt{2}}}{\left(\frac{1}{\sqrt{2}}\right)^3 + \left(\frac{1}{\sqrt{2}}\right)^3} = \frac{1}{2}\sqrt{2}\,a$	**B1** **1**	For correct value of *r*.**oe**
(iv)		**B1** **B1** **2**	Closed curve in 1st quadrant only, symmetrical about $\theta = \frac{1}{4}\pi$ Diagram showing $\theta = 0, \frac{1}{2}\pi$ tangential at *O*

5(i)	$x = \sin y \Rightarrow \frac{dx}{dy} = \cos y$ $\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$ $+\sqrt{\ }$ taken since $\sin^{-1} x$ has positive gradient	**M1** **A1** **B1** **3**	For implicit diffn to $\frac{dy}{dx} = \pm\frac{1}{\cos y}$ **oe** For using $\sin^2 y + \cos^2 y = 1$ to obtain **N.B. Answer given** For justifying + sign
(ii)	$f(0) = 0,\ f'(0) = 1$ $f''(x) = \frac{x}{\left(1-x^2\right)^{\frac{3}{2}}}$ $f'''(x) = \frac{\left(1-x^2\right)^{\frac{3}{2}} + 3x^2\left(1-x^2\right)^{\frac{1}{2}}}{\left(1-x^2\right)^3}$ $\Rightarrow f''(0) = 0,\ \ f'''(0) = 1$ $\Rightarrow \sin^{-1} x = x + \frac{1}{6}x^3$	**B1** **M1** **M1** **A1** **A1** **5**	For correct values Use of chain rule to differentiate $f'(x)$ Use of quotient or product rule to differentiate f ″ (0). For correct values **www, soi** For correct series (allow 3!) **www**
	Alternative Method: f(0) = 0, f'(0) = 1 $f'(x) = \frac{1}{\sqrt{1-x^2}} = \left(1-x^2\right)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{3}{8}x^4 + \dots$ $f''(x) = x + \frac{3}{2}x^3 + \dots$ $f'''(x) = 1 + \frac{9}{2}x^2 + \dots$ $\Rightarrow f'(0) = 1, f''(0) = 0, f'''(0) = 1$ $\Rightarrow \sin^{-1} x = x + \frac{1}{6}x^3$	**B1** **M1** **M1** **A1** **A1**	For correct values Correct use of binomial Differentiate twice Correct values Correct series
(iii)	$\left(\sin^{-1} x\right)\ln(1+x)$ $= \left(x + \frac{1}{6}x^3\right)\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right)$ $= x^2 - \frac{1}{2}x^3 + \frac{1}{2}x^4$	**B1ft** **M1** **A1** **A1** **4**	For terms in both series to at least x^3 f.t. from their **(ii)** multiplied together For multiplying terms to at least x^3 For correct series up to x^3 **www** For correct term in x^4 **www**

6(i)	$I_n = \int_0^1 x^n (1-x)^{\frac{3}{2}}\ \mathrm{d}x$	**M1**	For integrating by parts (correct way round)
	$= \left[-\frac{2}{5} x^n (1-x)^{\frac{5}{2}} \right]_0^1 + \frac{2}{5} n \int_0^1 x^{n-1} (1-x)^{\frac{5}{2}}\ \mathrm{d}x$	**A1**	For correct first stage
	$\Rightarrow I_n = \frac{2}{5} n \int_0^1 x^{n-1} (1-x)^{\frac{5}{2}}\ \mathrm{d}x$	**A1**	
	$\Rightarrow I_n = \frac{2}{5} n \int_0^1 x^{n-1} (1-x)(1-x)^{\frac{3}{2}}\ \mathrm{d}x$	**M1**	For splitting $(1-x)^{5/2}$ suitably
	$\Rightarrow I_n = \frac{2}{5} n I_{n-1} - \frac{2}{5} n I_n$	**A1**	For obtaining correct relation between I_n and I_{n-1}
	$\Rightarrow I_n = \frac{2n}{2n+5} I_{n-1}$	**A1** **6**	For correct result (**N.B. answer given**)
(ii)	$I_0 = \left[-\frac{2}{5}(1-x)^{\frac{5}{2}} \right]_0^1 = \frac{2}{5}$	**M1**	For evaluating I_0 [*OR* I_1 by parts]
		M1	For using recurrence relation 3 [*OR* 2] times (may be combined together)
	$I_3 = \frac{6}{11} I_2 = \frac{6}{11} \times \frac{4}{9} I_1 = \frac{6}{11} \times \frac{4}{9} \times \frac{2}{7} I_0$	**A1**	For 3 [*OR* 2] correct fractions
	$I_3 = \frac{32}{1155}$	**A1** **4**	For correct exact result

7(i)		B1 B1 B1 B1 4	Both curves of the correct shape (ignore overlaps) and labelled gradient = 1 at $x = 0$ stated For asymptotes $y = \pm 1$ and $x = \pm 1$ (or on sketch) Sketch all correct
(ii)	$\int_0^k \tanh x \, dx = [\ln(\cosh x)]_0^k = \ln(\cosh k)$	M1 A1 2	For substituting limits into $\ln \cosh x$ For correct answer
(iii)	 Areas shown are equal: $x = \tanh k$ $\Rightarrow y = k$ $\Rightarrow \int_0^{\tanh k} \tanh^{-1} x \, dx$ = rectangle ($k \times \tanh k$)– **(ii)** $= k \tanh k - \ln(\cosh k)$	M1 A1 M1 A1 4	For consideration of areas For sufficient justification For subtraction from rectangle For correct answer **N.B. answer given** **Alternative:** Otherwise by parts, as $1 \times \tanh^{-1} x$ *OR* $1 \times \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$

PTO for alternative schemes

7(iii)	Alternative method 1 By parts: $I = \int_0^{\tanh k} \tanh^{-1} x \, dx$ $u = \tanh^{-1} x \qquad dv = dx$ $du = \frac{1}{1-x^2} dx \qquad v = x$ $\Rightarrow I = \left[x \tanh^{-1} x \right]_0^{\tanh k} - \int_0^{\tanh k} \frac{x}{1-x^2} \, dx$ $= k \tanh k + \frac{1}{2} \left[\ln(1-x^2) \right]_0^{\tanh k}$ $= k \tanh k + \frac{1}{2} \ln(1-\tanh^2 k)$ $= k \tanh k + \frac{1}{2} \ln(\text{sech}^2 k)$ $= k \tanh k + \ln(\text{sech} k)$	**M1** **A1** **M1** **A1**	For integrating by parts (correct way round) For getting this far Dealing with the resulting integral
	Alternative method 2 By substitution Let $y = \tanh^{-1} x \Rightarrow x = \tanh y$ $\Rightarrow dx = \text{sech}^2 y \, dy$ When $x = 0, \; y = 0$ When $x = \tanh k, \; y = k$ $\Rightarrow I = \int_0^{\tanh k} \tanh^{-1} x \, dx = \int_0^k y \text{sech}^2 y \, dy$ $u = y \quad dv = \text{sech}^2 y \, dy$ $du = dy \qquad v = \tanh y$ $\Rightarrow I = \left[y \tanh y \right]_0^k - \int_0^k \tanh y \, dy$ $= k \tanh k - \ln \cosh k$	**M1** **A1** **M1** **A1**	For substitution to obtain equivalent integral Correct so far For integration by parts (correct way round) Final answer

8(i)	$x=\cosh^2 u \Rightarrow \mathrm{d}u = 2\cosh u \sinh u\,\mathrm{d}u$	**B1**	For correct result
	$\int \sqrt{\frac{x}{x-1}}\,\mathrm{d}x = \int \frac{\cosh u}{\sinh u} 2\cosh u \sinh u\,\mathrm{d}u$	**M1**	For substituting throughout for x
	$= \int 2\cosh^2 u\,\mathrm{d}u$	**A1**	For correct simplified u integral
	$= \int (\cosh 2u + 1)\,\mathrm{d}u = \sinh u \cosh u + u$	**M1**	For attempt to integrate $\cosh^2 u$
		A1	For correct integration
	$= x^{\frac{1}{2}}(x-1)^{\frac{1}{2}} + \ln\left(x^{\frac{1}{2}} + (x-1)^{\frac{1}{2}}\right) \; (+c)$	**M1**	For substituting for u
		A1 **7**	For correct result **oe** as $\mathrm{f}(x)+\ln(\mathrm{g}(x))$
(ii)	$2\sqrt{3} + \ln(2+\sqrt{3})$	**B1** **1**	
(iii)	$V = (\pi)\int_1^4 \frac{x}{x-1}\,\mathrm{d}x = (\pi)\left[x + \ln(x-1)\right]_1^4$	**M1**	For attempt to find $\int \frac{x}{x-1}\,\mathrm{d}x$
		A1	For correct integration (ignore π)
	$V \to \infty$	**B1** **3**	For statement that volume is infinite (independent of M mark)

OCR (Oxford Cambridge and RSA Examinations)
1 Hills Road
Cambridge
CB1 2EU

OCR Customer Contact Centre

14 – 19 Qualifications (General)
Telephone: 01223 553998
Facsimile: 01223 552627
Email: general.qualifications@ocr.org.uk

www.ocr.org.uk

For staff training purposes and as part of our quality assurance programme your call may be recorded or monitored

Oxford Cambridge and RSA Examinations
is a Company Limited by Guarantee
Registered in England
Registered Office; 1 Hills Road, Cambridge, CB1 2EU
Registered Company Number: 3484466
OCR is an exempt Charity

OCR (Oxford Cambridge and RSA Examinations)
Head office
Telephone: 01223 552552
Facsimile: 01223 552553