

OXFORD CAMBRIDGE AND RSA EXAMINATIONS

**Advanced Subsidiary General Certificate of Education
Advanced General Certificate of Education**

MATHEMATICS

4727

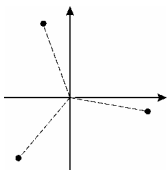
Further Pure Mathematics 3

MARK SCHEME

Specimen Paper

MAXIMUM MARK	72
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This mark scheme consists of 4 printed pages.

1 Integrating factor is $e^{\int -x^{-1} dx} = e^{-\ln x} = \frac{1}{x}$ $\frac{d}{dx}\left(\frac{y}{x}\right) = 1 \Rightarrow \frac{y}{x} = \int 1 dx \Rightarrow y = x^2 + cx$	M1 A1 M1 B1 A1	For finding integrating factor For correct simplified form For using integrating factor correctly For arbitrary constant introduced correctly For correct answer in required form 5 5
2 (i) b is the identity and so has order 1 $d * d = b$, so d has order 2 $a * a = c * c = d$, so a and c each have order 4 (ii) $\{b, d\}$ (iii) G is cyclic because it has an element of order 4 (iv) $b = 1, d = -1, a = i, c = -i$ (or vice versa for a, c)	B1 B1 B1 B1	For identifying b as the identity element For stating the order of d is 2 For both orders stated 3 1 1 1 6
3 (i) Normals are $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ Acute angle is $\cos^{-1}\left(\frac{ 2 - 4 - 2 }{3 \times 3}\right) \approx 64^\circ$ (ii) Direction of line is $(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \times (2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$, i.e. $-2\mathbf{i} + 5\mathbf{j} + 6\mathbf{k}$ $x - 2y + 2z = 1, 2x + 2y - z = 3 \Rightarrow 3x + z = 4$, so a common point is $(1, 1, 1)$, for example Hence line is $\mathbf{r} = \mathbf{i} + \mathbf{j} + \mathbf{k} + t(-2\mathbf{i} + 5\mathbf{j} + 6\mathbf{k})$	B1 M1 M1 A1 M1 A1	For identifying both normal vectors For using the scalar product of the normals For completely correct process for the angle For correct answer 4 For using vector product of normals For correct vector for $\mathbf{b}$ For complete method to find a suitable $\mathbf{a}$ For correct equation of line (Other methods are possible) 4 8
4 (i) $4((\sqrt{3}) - i) = 8e^{-\frac{1}{6}\pi i}$ (ii) One cube root is $2e^{-\frac{1}{18}\pi i}$ Others are found by multiplying by $e^{\pm\frac{2}{3}\pi i}$ Giving $2e^{\frac{11}{18}\pi i}$ and $2e^{-\frac{13}{18}\pi i}$ (iii)  The roots have equal modulus and args differing by $\frac{2}{3}\pi$, so adding them geometrically makes a closed equilateral triangle; i.e. sum is zero	B1 B1 B1√ M1 A1 A1 B1√ M1 A1	For $r = 8$ For $\theta = -\frac{1}{6}\pi$ 2 For modulus and argument both correct For multiplication by either cube root of 1 (or equivalent use of symmetry) For either one of these roots For both correct 4 For correct diagram from their (ii) For geometrical interpretation of addition For a correct proof (or via components, etc) 3 9

5	(i) $(\mathbf{i} - \mathbf{j} - 2\mathbf{k}) \times (-4\mathbf{i} - 14\mathbf{j} + 2\mathbf{k}) = -30\mathbf{i} + 6\mathbf{j} - 18\mathbf{k}$ So common perp is parallel to $5\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ $(5\mathbf{i} + \mathbf{j} + 5\mathbf{k}) - (\mathbf{i} + 11\mathbf{j} + 2\mathbf{k}) = 4\mathbf{i} - 10\mathbf{j} + 3\mathbf{k}$ $d = \frac{ (4\mathbf{i} - 10\mathbf{j} + 3\mathbf{k}) \cdot (5\mathbf{i} - \mathbf{j} + 3\mathbf{k}) }{ 5\mathbf{i} - \mathbf{j} + 3\mathbf{k} } = \frac{39}{\sqrt{35}}$	M1 A1 B1 M1 A1	For vector product of direction vectors For correct vector for common perp For calculating the difference of positions For calculation of the projection 5 For correct exact answer
(ii)	Normal vector for plane is $5\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ Point on plane is $5\mathbf{i} + \mathbf{j} + 5\mathbf{k}$ Equation is $5x - y + 3z = 25 - 1 + 15$ i.e. $5x - y + 3z = 39$	B1✓ B1 M1 A1	For stating or using the normal vector For using any point of l_1 For using relevant direction and point 4 For a correct equation
6	(i) $\mathbf{AQ} = \mathbf{QA} \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ i.e. $\begin{pmatrix} a & a+b \\ c & c+d \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ c & d \end{pmatrix}$ Hence $a = a + c$ and $a + b = b + d$ i.e. $c = 0$ and $d = a$	M1 A1 M1 A1	For considering $\mathbf{AQ} = \mathbf{QA}$ with general $\mathbf{A}$ For correct simplified equation For equating corresponding entries 4 For complete proof
(ii)	To be non-singular, $a \neq 0$	B1	1 For stating that a is non-zero
(iii)	Identity is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ as usual, since this is in S Inverse of $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ is $\begin{pmatrix} 1/a & -b/a^2 \\ 0 & 1/a \end{pmatrix}$, as $a \neq 0$ $\begin{pmatrix} a_1 & b_1 \\ 0 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 & a_1 b_2 + b_1 a_2 \\ 0 & a_1 a_2 \end{pmatrix}$ This is in S, since $a_1 a_2 \neq 0$, so all necessary group properties are shown	B1 B1 B1 M1 A1	For justifying the identity correctly For statement of correct inverse For justification via non-zero a For considering a general product 5 For complete proof
7	(i) $z^n = \cos n\theta + i \sin n\theta$ $z^{-n} = \cos n\theta - i \sin n\theta$, hence $z^n + z^{-n} = 2 \cos n\theta$	B1 B1	For applying de Moivre's theorem 2 For complete proof
(ii)	$2^6 \cos^6 \theta = (z + z^{-1})^6$ $= (z^6 + z^{-6}) + 6(z^4 + z^{-4}) + 15(z^2 + z^{-2}) + 20$ $= 2 \cos 6\theta + 12 \cos 4\theta + 30 \cos 2\theta + 20$ Hence $\cos^6 \theta = \frac{1}{32} (\cos 6\theta + 6 \cos 4\theta + 15 \cos 2\theta + 10)$ Integral is $\frac{1}{32} \left[\frac{1}{6} \sin 6\theta + \frac{3}{2} \sin 4\theta + \frac{15}{2} \sin 2\theta + 10\theta \right]_0^{\frac{1}{3}\pi}$ $= \frac{1}{32} \left(\frac{1}{6} \times 0 + \frac{3}{2} \times \left(-\frac{1}{2}\sqrt{3}\right) + \frac{15}{2} \times \left(\frac{1}{2}\sqrt{3}\right) + 10 \times \frac{1}{3}\pi \right)$ $= \frac{1}{32} \left(3\sqrt{3} + \frac{10}{3}\pi \right)$	M1 M1 A1 A1 M1 A1✓ M1 A1	For considering $(z + z^{-1})^6$ For expanding and grouping terms For correct substitution of multiple angles For correct answer For integrating multiple angle expression For correct terms For use of limits 8 For correct answer

8	(i) $y = kx^2 e^{-2x} \Rightarrow y' = 2kx e^{-2x} - 2kx^2 e^{-2x}$ and $y'' = 2k e^{-2x} - 8kx e^{-2x} + 4kx^2 e^{-2x}$ $(2k - 8kx + 4kx^2 + 8kx - 8kx^2 + 4kx^2) e^{-2x} \equiv 2e^{-2x}$ Hence $k = 1$	M1 A1 M1 A1	For differentiation at least once For both y' and y'' correct For substituting completely in D.E. For correct value of k
(ii)	Auxiliary equation is $m^2 + 4m + 4 = 0 \Rightarrow m = -2$ Hence C.F. is $(A + Bx)e^{-2x}$ G.S. is $y = (A + Bx)e^{-2x} + x^2 e^{-2x}$ $x = 0, y = 1 \Rightarrow 1 = A$ $y' = Be^{-2x} - 2(A + Bx)e^{-2x} + 2xe^{-2x} - 2x^2 e^{-2x}$ $x = 0, y' = 0 \Rightarrow 0 = B - 2A \Rightarrow B = 2$ Hence solution is $y = (1 + x)^2 e^{-2x}$	B1 B1 B1✓ M1 M1 M1 A1	For correct repeated root For correct form of C.F. For sum of C.F. and P.I. For using given values of x and y in G.S. For differentiating the G.S. For using given values of x and y' in G.S. For correct answer
(iii)	$\frac{d^2 y}{dx^2} = 2 - 4 = -2$ when $x = 0$ Hence $(0, 1)$ is a maximum point $\frac{dy}{dx} = 2(1 + x)e^{-2x} - 2(1 + x)^2 e^{-2x} = -2x(1 + x)e^{-2x}$, so there are no turning points for $x > 0$ Hence $0 < y \leq 1$, since $y \rightarrow 0$ as $x \rightarrow \infty$	B1 B1 M1 A1	For correct value -2 For statement of maximum at $x = 0$ For investigation of turning points, or equiv For complete proof of given result